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Ordinary Dichotomy and Perturbations of the Impulse Matrices of Linear Impulsive Differential Equation

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It is proved that the ordinary dichotomy is preserved under perturbations of the impulse matrices of linear impulsive differential equations.

1. INTRODUCTION

Recently the dichotomies for ordinary differential equations have been investigated by many authors (Coppel, 1978; Elaydi and Hájek, 1985, 1987, 1988, and to appear; Palmer, 1977, 1979*a,b*, 1982*a,b*, 1984*a,b*, 1987*a,b*, 1988; Sacker and Sell, 1974, 1976*a,b*, 1978). In Milev and Bainov (to appear) we first studied the dichotomies for linear impulsive differential equations. In the present paper we consider one of the important properties of the ordinary dichotomy for impulsive differential equations, namely that it is preserved under perturbations of the impulse matrices.

2. PRELIMINARY NOTES

Let $t_0 < t_1 < \cdots < t_i < \cdots$, $\lim t_i = \infty$ as $i \to \infty$, be a given sequence of real numbers. Consider the linear differential equation with impulses at fixed times

$$\frac{dx}{dt} = A(t)x, \qquad t \neq t_i$$
$$x(t_i + 0) = B_i x(t_i), \qquad i = 1, 2, \ldots \tag{1}$$

where the $(n \times n)$ coefficient matrix $A(t)$ is piecewise continuous in the interval $[t_0, +\infty)$ with points of discontinuity of the first kind at $t = t_i$ and

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the impulse matrices B_i, $i = 1, 2, \ldots$, are constant. The underlying vector space E is $\mathbb{R}^n$ or $\mathbb{C}^n$.

Remark 1. For $t \in [t_i + 0, t_{i+1}]$ the fundamental matrix $X(t)$ of equation (1) admits the representation

$$X(t) = U(t)U^{-1}(t_i+0)B_iU(t_i)U^{-1}(t_{i-1}+0)B_{i-1}\ldots B_1U(t_1)U^{-1}(t_0)$$

where $U(t)$ is the fundamental matrix of the equation $dx/dt = A(t)x$. The matrix $X(t)$ is continuously differentiable for $t \neq t_i$ with points of discontinuity of the first kind at $t = t_i$, i.e., $X(t_i+0) = B_iX(t_i)$. The fundamental matrix $X(t)$ is invertible if and only if the impulse matrices B_i, $i = 1, 2, \ldots$, are nonsingular.

Together with equation (1), consider its perturbed equation obtained by a perturbation of the impulse matrices

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, \qquad t \neq t_i \\ x(t_i+0) &= (B_i + \tilde{B}_i)x(t_i), \qquad i = 1, 2, \ldots \end{aligned} \tag{2}$$

where the matrices $\tilde{B}_i$, $i = 1, 2, \ldots$, are constant.

Let τ_0 be a fixed real number, $\tau_0 \geq t_0$.

Definition 1 (Milev and Bainov, to appear). The subspace Y of the underlying vector space E induces an ordinary dichotomy of the solutions of equation (1) on the interval $[\tau_0, +\infty)$ if for some subspace Z supplementary to Y there exists a constant N such that all solutions x, y, z of equation (1) with $x = y + z$, $y(\tau_0) \in Y$, and $z(\tau_0) \in Z$ satisfy the conditions

$$\begin{aligned} |y(t)| &\leq N|x(s)| \qquad \text{for} \quad t \geq s \geq \tau_0 \\ |z(t)| &\leq N|x(s)| \qquad \text{for} \quad s \geq t \geq \tau_0 \end{aligned} \tag{3}$$

When the fundamental matrix $X(t)$ is invertible, Definition 1 can be written as follows:

The subspace Y of the underlying vector space E induces an ordinary dichotomy of the solutions of equation (1) on the interval $[\tau_0, +\infty)$ if for some projector P ($P^2 = P$) with range $R(P) = Y$ (the kernel of P coincides with Z) there exists a constant N such that

$$\begin{aligned} |X(t)X^{-1}(\tau_0)PX(\tau_0)X^{-1}(s)| &\leq N \qquad \text{for} \quad t \geq s \geq \tau_0 \\ |X(t)X^{-1}(\tau_0)(I-P)X(\tau_0)X^{-1}(s)| &\leq N \qquad \text{for} \quad s \geq t \geq \tau_0 \end{aligned} \tag{4}$$

where I stands for the unit matrix.

Definition 2. Let P ($P^2 = P$) be a projector. The function

$$G(t, s) = \begin{cases} X(t)PX^{-1}(s) & \text{for } t \geq s \geq t_0 \\ X(t)(P - I)X^{-1}(s) & \text{for } s > t \geq t_0 \end{cases}$$

will be called the Green's function for equation (1).

We shall use the following properties of the Green's function, which can be verified immediately:

$$\frac{\partial G(t, s)}{\partial t} = A(t)G(t, s), \qquad t \neq s \tag{5}$$

$$G(t_i + 0, t) = B_i G(t_i, t), \qquad t \neq t_i, \quad i = 1, 2, \ldots \tag{6}$$

$$G(t_i + 0, t_i + 0) = B_i G(t_i, t_i + 0) + I, \qquad i = 1, 2, \ldots \tag{7}$$

3. MAIN RESULTS

Theorem 1. Let the impulse matrices B_i, $i = 1, 2, \ldots$, of equation (1) be nonsingular and let the subspace Y induce an ordinary dichotomy of the solutions of equation (1) on the interval $[t_0, +\infty)$ with a projector P and constant N. If

$$\sum_{i=1}^{\infty} |\tilde{B}_i| = K < \frac{1}{N(2N+1)} \tag{8}$$

then the perturbed equation (2) also has an ordinary dichotomy on the interval $[t_0, +\infty)$.

Proof. Let $X(t)$ be the fundamental matrix of equation (1) for which $X(t_0) = I$. The bounded solutions $y(t)$ of equation (2) are just the bounded solutions of the equation

$$y(t) = X(t)\eta + \sum_{j=1}^{\infty} G(t, t_j + 0)\tilde{B}_j y(t_j), \qquad \eta \in Y \tag{9}$$

since for $t \neq t_i$, $dy(t)/dt = A(t)y(t)$ and for $t = t_i$,

$$\begin{aligned} y(t_i + 0) &= X(t_i + 0)\eta + \sum_{j=1}^{\infty} G(t_i + 0, t_j + 0)\tilde{B}_j y(t_j) \\ &= B_i X(t_i)\eta + \sum_{j=1}^{\infty} B_i G(t_i, t_j + 0)\tilde{B}_j y(t_j) + \tilde{B}_i y(t_i) \\ &= (B_i + \tilde{B}_i) y(t_i) \end{aligned}$$

Denote by H the Banach space of all bounded, piecewise-continuous, vector-valued functions $y(t)$ in the interval $[t_0, +\infty)$ with points of discontinuity of the first kind at $t = t_i$, $y(t_i) = y(t_i - 0)$, $i = 1, 2, \ldots$, and with a norm $\|y\| = \sup_{t \geq t_0} |y(t)|$.

The linear operator

$$Ly(t) = \sum_{j=1}^{\infty} G(t, t_j + 0) \tilde{B}_j y(t_j)$$

maps H into itself since

$$|Ly(t)| \leq \sum_{j=1}^{\infty} |G(t, t_j + 0)| |\tilde{B}_j| |y(t_j)| \leq NK \|y\|$$

This implies that $|L| \leq NK < 1$ and by the contraction principle, equation (9) for any $\eta \in Y$ has exactly one solution $y \in H$ which depends linearly on η, i.e., $y(t) = F(t)\eta$, where $F(t)$ is a bounded matrix. By (9)

$$y = X(t)\eta + Ly(t) = X(t)PX^{-1}(t_0)\eta + Ly(t)$$

since $X(t_0) = I$ and $P\eta = \eta$. Hence

$$\|y\| \leq N|\eta| + |L| \|y\| \leq N|\eta| + NK \|y\|$$

i.e.,

$$\|y\| \leq \frac{N}{1 - NK} |\eta| \qquad \text{and} \qquad |F(t)| \leq \frac{N}{1 - NK}$$

Let $\tilde{Y}$ be a subspace of E consisting of the initial values $y(t_0)$ of the bounded solutions of equation (2),

$$\begin{aligned} y(t_0) &= \eta + \sum_{j=1}^{\infty} G(t_0, t_j + 0) \tilde{B}_j y(t_j) \\ &= \eta - (I - P) \sum_{j=1}^{\infty} X^{-1}(t_j + 0) \tilde{B}_j F(t_j) \eta \\ &= (I - (I - P)QP)\eta \end{aligned}$$

where

$$Q = (I - P) \sum_{j=1}^{\infty} X^{-1}(t_j + 0) \tilde{B}_j F(t_j) P$$

$$|Q| \leq N \cdot K \cdot \frac{N}{1 - NK} \cdot |P|$$

Hence the operator $I-(I-P)QP$ maps the subspace Y onto $\tilde{Y}$. This operator has a bounded inverse on $I+(I-P)QP$. The operator

$$\tilde{P}=(I-(I-P)QP)P(I-(I-P)QP)^{-1}=P-(I-P)QP$$

is a projector with range $R(\tilde{P})=\tilde{Y}$. The supplementary projector $I-\tilde{P}=(I-P)(I+QP)$ has a range $R(I-\tilde{P})=Z$.

First we shall estimate the solutions starting from $\tilde{Y}$. Let $s\in[t_k+0,\ t_{k+1}]$. By (9)

$$\begin{aligned}\eta&=X^{-1}(s)y(s)-X^{-1}(s)\sum_{j=1}^{\infty}G(s,t_j+0)\tilde{B}_jy(t_j)\\&=X^{-1}(s)y(s)-\sum_{j=1}^{k}PX^{-1}(t_j+0)\tilde{B}_jy(t_j)\\&\quad-\sum_{j=k+1}^{\infty}(P-I)X^{-1}(t_j+0)\tilde{B}_jy(t_j)\\&=PX^{-1}(s)y(s)-\sum_{j=1}^{k}PX^{-1}(t_j+0)\tilde{B}_jy(t_j)\end{aligned}$$

i.e.,

$$\begin{aligned}y(t)&=X(t)\eta+\sum_{j=1}^{\infty}G(t,t_j+0)\tilde{B}_jy(t_j)\\&=X(t)PX^{-1}(s)y(s)+\sum_{j=k+1}^{\infty}G(t,t_j+0)\tilde{B}_jy(t_j)\end{aligned}$$

Hence for $t\geq s$,

$$|y(t)|\leq N|y(s)|+N\sum_{j=k+1}^{\infty}|\tilde{B}_j||y(t_j)|$$

Let us fix s and set $N|y(s)|=\alpha$. The cone of the nonnegative piecewise continuous functions $\varphi(t)=|y(t)|$ is invariant with respect to the linear operator

$$T\varphi(t)=N\sum_{j=k+1}^{\infty}|\tilde{B}_j|\varphi(t_j)$$

i.e., if $\varphi(t)\leq\psi(t)$, then $T\varphi(t)\leq T\psi(t)$. Hence from $|y(t)|\leq\alpha+T|y(t)|$, we obtain that

$$T|y(t)|\leq T\alpha+T^2|y(t)|=NK\alpha+NKT|y(t)|$$

i.e.

$$|y(t)| \le \alpha + T|y(t)| \le \alpha + \frac{NK\alpha}{1-NK} = \frac{\alpha}{1-NK}$$

Hence for $t \ge s$,

$$|y(t)| \le \frac{N}{1-NK}|y(s)| \tag{10}$$

Let $z(t)$ be a solution with initial condition $z(t_0) \in Z$ and let $t \in [t_m+0, t_{m+1}]$. Then $z(t)$ is a solution of the equation

$$z(t) = X(t)z(t_0) + \sum_{j=1}^{m} X(t)X^{-1}(t_j+0)\tilde{B}_j z(t_j)$$

since $dz(t)/dt = A(t)z(t)$ and

$$\begin{aligned} z(t_i+0) &= X(t_i+0)z(t_0) + \sum_{j=1}^{i} X(t_i+0)X^{-1}(t_j+0)\tilde{B}_j z(t_j) \\ &= B_i X(t_i)z(t_0) + \sum_{j=1}^{i-1} B_i X(t_i)X^{-1}(t_j+0)\tilde{B}_j z(t_j) + \tilde{B}_i z(t_i) \\ &= B_i z(t_i) + \tilde{B}_i z(t_i) = (B_i + \tilde{B}_i)z(t_i) \end{aligned}$$

Let $s \in [t_k+0, t_{k+1}]$. From the formula

$$z(s) = X(s)z(t_0) + \sum_{j=1}^{k} X(s)X^{-1}(t_j+0)\tilde{B}_j z(t_j)$$

we express $z(t_0)$ and in view of $(I-P)z(t_0) = z(t_0)$ for $t < s$ we obtain

$$\begin{aligned} z(t) &= X(t)(I-P)X^{-1}(s)z(s) - \sum_{j=1}^{k} X(t)(I-P)X^{-1}(t_j+0)\tilde{B}_j z(t_j) \\ &\quad + \sum_{j=1}^{m} X(t)X^{-1}(t_j+0)\tilde{B}_j z(t_j) \\ &= X(t)(I-P)X^{-1}(s)z(s) + \sum_{j=1}^{m} X(t)PX^{-1}(t_j+0)\tilde{B}_j z(t_j) \\ &\quad + \sum_{m<j\le k} X(t)(I-P)X^{-1}(t_j+0)\tilde{B}_j z(t_j) \end{aligned}$$

and deduce the inequality

$$|z(t)| \le N|z(s)| + N\sum_{j=1}^{k} |\tilde{B}_j|\,|z(t_j)|$$

The linear operator

$$T_1\varphi(t)=N\sum_{j=1}^{k}|\tilde{B}_j|\varphi(t_j)$$

is monotone and, as for (10), we obtain that

$$|z(t)|\leq\frac{N}{1-NK}|z(s)| \qquad \text{for} \quad t\leq s \tag{11}$$

Let $x(t)=y(t)+z(t)$ be an arbitrary solution of equation (2) and let $s\in[t_k+0,\ t_{k+1}]$. From the formula

$$x(s)=X(s)x(t_0)+\sum_{j=1}^{k}X(s)X^{-1}(t_j+0)\tilde{B}_jx(t_j)$$

we express $x(t_0)$ and in view of (9) we obtain

$$\begin{aligned}
y(s)&=X(s)Px(t_0)+\sum_{j=1}^{\infty}G(s,\ t_j+0)\tilde{B}_jy(t_j)\\
&=X(s)PX^{-1}(s)x(s)-\sum_{j=1}^{k}X(s)PX^{-1}(t_j+0)\tilde{B}_jx(t_j)\\
&\quad+\sum_{j=1}^{k}X(s)PX^{-1}(t_j+0)\tilde{B}_jy(t_j)\\
&\quad+\sum_{j=k+1}^{\infty}X(s)(I-P)X^{-1}(t_j+0)\tilde{B}_jy(t_j)\\
&=X(s)PX^{-1}(s)x(s)-\sum_{j=1}^{k}X(s)PX^{-1}(t_j+0)\tilde{B}_jz(t_j)\\
&\quad+\sum_{j=k+1}^{\infty}X(s)(I-P)X^{-1}(t_j+0)\tilde{B}_jy(t_j)
\end{aligned}$$

In view of (10) and (11) we deduce the inequality

$$\begin{aligned}
|y(s)|&\leq N|x(s)|+N\sum_{j=1}^{k}|\tilde{B}_j||z(t_j)|+N\sum_{j=k+1}^{\infty}|\tilde{B}_j||y(t_j)|\\
&\leq N|x(s)|+\frac{N^2K}{1-NK}|z(s)|+\frac{N^2K}{1-NK}|y(s)|\\
&\leq\frac{N}{1-NK}|x(s)|+\frac{2N^2K}{1-NK}|y(s)|
\end{aligned}$$

Hence

$$|y(s)| \le \frac{N}{1-NK-2N^2K}|x(s)|$$

By (10)-(12) for $t \ge s$

$$|y(t)| \le \frac{N^2}{(1-NK)(1-NK-2N^2K)}|x(s)|$$

and for $t \le s$

$$|z(t)| \le \frac{N}{1-NK}|z(s)| \le \frac{N}{1-NK}(|x(s)|+|y(s)|)$$
$$\le \frac{N(1+N-NK-2N^2K)}{(1-NK)(1-NK-2N^2K)}|x(s)|$$

i.e.,

$$|y(t)| \le N_1|x(s)| \qquad \text{for} \quad t \ge s \ge t_0$$
$$|z(t)| \le N_1|x(s)| \qquad \text{for} \quad s \ge t \ge t_0$$

where

$$N_1 = \frac{N(1+N-NK-2N^2K)}{(1-NK)(1-NK-2N^2K)} > 0$$

Hence the perturbed equation (2) has an ordinary dichotomy on the interval $[t_0, +\infty)$. ■

Corollary 1. If the conditions of Theorem 1 hold, then the perturbed equation (2) has an ordinary dichotomy on each subinterval $[\tau_0, +\infty)$, $\tau_0 \ge t_0$, as well.

Proof. Since conditions (4) are valid in the interval $[\tau_0, +\infty)$ as well, then equation (1) has an ordinary dichotomy on the interval $[\tau_0, +\infty)$, moreover, condition (8) holds. Then, by Theorem 1, the perturbed equation (2) also has an ordinary dichotomy on the interval $[\tau_0, +\infty)$. ■

Remark 2 (Milev and Bainov, to appear). In the classical case, if the linear differential equation (without impulse effect) has an ordinary dichotomy on the interval $[t_0, +\infty)$, then it has an ordinary dichotomy on each subinterval $[\tau_0, +\infty)$, $\tau_0 \ge t_0$, as well. For the linear impulsive differential equations the following phenomenon is observed. The equation may have an ordinary dichotomy on the interval $[t_k+0, +\infty)$ and have no dichotomy on the subinterval $[t_{k+1}, +\infty)$. We illustrate this by the following example.

Example 1. Let $t_i = i$, $i = 0, 1, 2, \ldots$, and consider the linear impulsive differential equation

$$\frac{dx}{dt} = Ax, \qquad t \neq t_i$$
$$x(t_i + 0) = B_i x(t_i), \qquad i = 1, 2, \ldots$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_i = I \quad \text{for} \quad i \geq 2$$

It is verified directly that the equation has an ordinary dichotomy on the interval $[0, +\infty)$ since the impulse at the moment t_1 crumples the "inappropriate" solutions. The equation has no dichotomy on any of the subintervals $[\tau, +\infty)$, $\tau > 1$, because there the problem coincides with the classical one and the eigenvalues of the matrix A with a zero real part are not semisimple.

Lemma 1 (Milev and Bainov, to appear). Let τ_0 and τ be fixed real numbers in the interval $[t_0, +\infty)$ and let the impulse matrices B_i, $i = 1, 2, \ldots$, of equation (1) be nonsingular. If equation (1) has an ordinary dichotomy on the interval $[\tau_0, +\infty)$ with a projector $P(\tau_0)$, then it has an ordinary dichotomy on the interval $[\tau, +\infty)$ as well with a projector $P(\tau) = X(\tau)X^{-1}(\tau_0)P(\tau_0)X(\tau_0)X^{-1}(\tau)$.

Proof. For $\tau \geq \tau_0$ the assertion is obvious, since for $t \geq s \geq \tau$

$$X(t)X^{-1}(\tau)P(\tau)X(\tau)X^{-1}(s) = X(t)X^{-1}(\tau_0)P(\tau_0)X(\tau_0)X^{-1}(s)$$

and conditions (4) are fulfilled.

Let $\tau < \tau_0$. By the Gronwall–Bellman inequality for $\tau_1, \tau_2 \in [t_m + 0, t_{m+1}]$

$$|U(\tau_1)U^{-1}(\tau_2)| \leq \exp\left|\int_{\tau_1}^{\tau_2} |A(\theta)|\, d\theta\right|$$

Let $t > s$ and $t \in [t_i + 0, t_{i+1}]$ and $s \in [t_j + 0, t_{j+1}]$. Then the fundamental matrix $X(t)$ has the form

$$\begin{aligned} X(t) = {} & U(t)U^{-1}(t_i + 0)B_i U(t_i)U^{-1}(t_{i-1} + 0) \\ & \times B_{i-1} \ldots B_{j+1} U(t_j + 0)U^{-1}(s)X(s) \end{aligned}$$

Hence

$$\begin{aligned} |X(t)X^{-1}(s)| & \leq K_i K_{i-1} \ldots K_{j+1} \exp \int_s^t |A(\theta)|\, d\theta \\ & \leq K_i K_{i-1} \ldots K_{j+1} \exp \int_\tau^{\tau_0} |A(\theta)|\, d\theta = K, \qquad t, s \in [\tau, \tau_0] \end{aligned}$$

where $K_\nu = \max(|B_\nu|, |B_\nu^{-1}|) \geq 1$. In the same way we prove that $|X(s)X^{-1}(t)| \leq K$. Hence for any $t, s \in [\tau, \tau_0]$ the inequality $|X(t)X^{-1}(s)| \leq K$ holds. Without loss of generality, let $K \geq 1$.

If $\tau \leq s < \tau_0 \leq t$, then

$$|X(t)X^{-1}(\tau)P(\tau)X(\tau)X^{-1}(s)| = |X(t)X^{-1}(\tau_0)P(\tau_0)X(\tau_0)X^{-1}(\tau_0)X(\tau_0)X^{-1}(s)| \leq NK$$

If $\tau \leq s \leq t < \tau_0$, then

$$|X(t)X^{-1}(\tau)P(\tau)X(\tau)X^{-1}(s)| \leq |X(t)X^{-1}(\tau_0)|N|X(\tau_0)X^{-1}(s)| \leq NK^2$$

Hence for $t \geq s \geq \tau$,

$$|X(t)X^{-1}(\tau)P(\tau)X(\tau)X^{-1}(s)| \leq NK^2$$

In the same way it is verified that for $s \geq t \geq \tau$,

$$|X(t)X^{-1}(\tau)(I - P(\tau))X(\tau)X^{-1}(s)| \leq NK^2 \quad \blacksquare$$

As a consequence of Theorem 1 and Lemma 1, we obtain the following assertion.

Corollary 2. Let the impulse matrices B_i and $B_i + \tilde{B}_i$, $i = 1, 2, \ldots$, be nonsingular and let equation (1) have an ordinary dichotomy on the interval $[t_0, +\infty)$. If

$$\sum_{i=1}^{\infty} |\tilde{B}_i| < \infty$$

then the perturbed equation (2) also has an ordinary dichotomy on the interval $[t_0, +\infty)$.

Proof. There exists a number k so large that

$$\sum_{i=k+1}^{\infty} |\tilde{B}_i| < \frac{1}{N(2N+1)}$$

Since the impulse matrices B_i, $i = 1, 2, \ldots$, are nonsingular, then equation (1) has an ordinary dichotomy with the same constant N on the interval $[t_k + 0, +\infty)$ as well. Then by Theorem 1 the perturbed equation (2) also has an ordinary dichotomy on the interval $[t_k + 0, +\infty)$ and by Lemma 1, equation (2) has an ordinary dichotomy on the interval $[t_0, +\infty)$ as well. $\blacksquare$

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